Finding Optimal Refueling Policies in Transportation Networks

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Abstract. We study the combinatorial properties of optimal refueling policies, which specify the transportation paths and the refueling operations along the paths to minimize the total transportation costs between vertices. The insight into the structure of optimal refueling policies leads to an elegant reduction of the problem of finding optimal refueling policies into the classical shortest path problem, which ends in simple and more efficient algorithms for finding optimal refueling policies.

1 Introduction

A vehicle refueling policy is a path in a transportation network together with the series of refueling operations when passing through the vertices on the path to reach a destination vertex from a starting vertex in the network while always maintaining the fuel level between a lower limit L and an upper limit U throughout the entire process. An optimal refueling policy minimizes the total fuel cost to reach the destination vertex given an initial fuel level and a required minimal final fuel level. Different from the shortest path problem, fuel prices at the vertices must be considered in addition to the distances between vertices, and there are situations in which optimal refueling policies involve non-simple paths. Optimal refueling polices can be determined by solving mixed integer programs [6]. Lin et al. [4] gives a linear-time algorithm for determining an optimal refueling policy given a fixed path and relate the problem to the bounded-inventory economic lot-sizing problems [1] [5]. Given a general transportation network of n vertices and the all-pairs shortest-distance information, Khuller et al. [3] provides algorithms for (i) finding all-pairs optimal refueling policies given the initial fuel levels at the vertices in $O(n^4)$ time in general or in $O(n^3 k * \min(k, \log n))$ time if the vehicle is constrained to use at most k refueling stops respectively and (ii) finding an optimal refueling policy for a given pair of vertices and an initial fuel level in $O(n^3)$ time in general or in $O(n^2 k \log n)$ time with the k-stop constraint. Both of Lin and Khuller et al. assume the required minimal final fuel level reaching the destination to be the same as the lower fuel limit L.

In this paper, we first analyze the combinatorial structure of optimal $[L, L]$ refueling policies, which are refueling policies required to begin with a fuel level at the lower limit L and ending with a fuel level at least at the lower limit L. We prove that finding optimal $[L, L]$ vehicle refueling policies in a transportation

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network G is equivalent to finding shortest paths in an optimal transition graph derived from G, which is essentially a finite automaton modelling all possible optimal transitions between the upper and the lower limits of fuel level between vertices together with a distance measure representing the optimal transition costs. This leads to simple algorithms that can determine optimal $[L, L]$ refueling policies for all pairs of vertices in a network of n vertices in $O(n^3)$ time or in $O(n^3 \log k)$ time with the additional k-stop constraint. With the all-pairs optimal (k-stop-bounded) $[L, L]$ vehicle refueling policies determined, we can then (i) determine all-pairs optimal (k-stop-bounded) refueling policies given various initial fuel levels at the vertices and ending with a fuel level at least at the lower limit L in $O(n^3)$ time, and (ii) determine an optimal refueling policy given a pair of vertices, an initial fuel level, and a required minimal final fuel level in $O(n^2)$ time.

2 The Optimal Refueling Policy Problems

Consider a vehicle with a fixed fuel tank capacity operates in a transportation network delivering commodities between pairs of locations in the network. The vehicle needs to refuel at fuel stations to maintain a minimum level of fuel in the fuel tank all the time. Given a pair of locations s and t together with an initial fuel level starting at s and a required fuel level when arriving at t, we would like to determine an optimal refueling policy specifying a path from s to t, the fuel stations on the path to stop for refueling, and the amounts of fuel to add in the fuel stations to minimizes the total refueling cost.

Definition 1. (Vehicle-network instances). An vehicle-network instance is a six-tuple $\langle L, U, V, A, X, P \rangle$ where

- $L, U \in \mathbb{Z}^+ \cup \{0\}$, $0 \le L < U$, L is the minimum fuel level the vehicle needs to maintain all the time while the minimum fuel level U is simply the full tank capacity of the vehicle,
- V and A together form a directed graph $G = (V, A)$ modelling a transportation network where V is the set of vertices representing points of interest in the network including locations of suppliers, customers, cities, and fuel stations in the transportation network, while A is the set of directed edges with each directed edge (u, v) in A representing a transportation link from vertex u to vertex v that requires no more than $U - L$ amount of fuel,
- $X : V \times V \to [0, U - L] \cup \{\infty\}$ is the fuel consumption function where $X(u, v)$ equals the amount of fuel consumed to reach vertex v from vertex u for $(u, v) \in A$, $X(u, u) = 0$ for $u \in V$, and $X(u, v) = \infty$ for $(u, v) \notin A$ when $u \ne v$, and in the following of the paper we use $\hat{X}(u, v)$ to denote the shortest distance from u to v with X as the underlying distance measure over the directed graph $G = (V, A)$, and
- $P : V \to \mathbb{Z}^+ \cup \{0, \infty\}$ is the fuel price function where $P(u)$ denotes the unit fuel price charged in vertex u and $P(u) = \infty$ if u does not have a fuel station.

Definition 2. (Vehicle refueling policies). Given a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$, a $[l_s, l_t]$ refueling policy from vertex s to vertex t is a sequence of vertex-and-refueling-amount pairs $\pi_{s\to t}^{l_s,l_t} = \langle (v_0, f_0), \ldots, (v_m, f_m)\rangle$ where $m \geq 0$, $\forall i \in [0, m]$ $v_i \in V$, $v_0 = s$, $v_m = t$, f_i, l_s and l_t are integers, $L \leq l_s, l_t \leq U$, and $0 \leq f_i \leq U - L$ such that

- $\forall i \in [0, m-1]$ $(v_i, v_{i+1}) \in A$ and $\langle v_0, \ldots, v_m\rangle$ is a path from s to t,
- $\forall i \in [0, m]$, $FE_\pi(v_i) =$ $l_s + \sum_{1\leq j<i} f_j - \sum_{1\leq j<i} X(v_j, v_{j+1}) \geq L$,
- $\forall i \in [0, m]$, $FL_\pi(v_i) =$ $l_s + \sum_{1\leq j\leq i} f_j - \sum_{1\leq j<i} X(v_j, v_{j+1}) \leq U$, and
- $FL_\pi(v_m) =$ $l_s + \sum_{1\leq j\leq m} f_j - \sum_{1\leq j<m} X(v_j, v_{j+1}) \geq l_t$

where l_s and l_t are the initial fuel level and the required minimal final fuel level at vertices s and t respectively, f_i is the amount of fuel to purchase at vertex v_i, $FE_\pi(v_i)$ denotes the fuel level when entering the ith vertex v_i on the path, and $FL_\pi(v_i)$ denotes the fuel level when leaving v_i or when finally ending at v_m when $i = m$.

Remark: A $[l_s, l_t]$ refueling policy from vertex s to vertex t represents a feasible refueling solution along an operational path $\langle v_0, \ldots, v_m\rangle$ of length m, allowing the vehicle to progressively move from vertex v_i to vertex v_{i+1} for $0 \leq i < m$, starting with an initial fuel level l_s at the starting vertex $s = v_0$, stopping at vertex v_i to purchase f_i units of fuel if f_i is not zero, ending in the destination vertex $t = v_m$ with a final fuel level of l_t or more in the end. The refueling policy ensures in the process it never reaches a fuel level lower than the lower limit L or higher than upper limit U by keeping $FE_\pi(v_i)$, the fuel level when entering the ith vertex v_i on the operation path, to at least L and keeping $FL_\pi(v_i)$, the fuel level when leaving v_i (or when settling down at v_m when $i = m$) to at most U. Note that $\langle v_0, \ldots, v_m\rangle$ is not necessarily a simple path since the vehicle may repeatedly leave a vertex v_i to refuel at other vertices with lower fuel prices and come back to v_i later with a higher fuel level needed to proceed with remainder of the path. A graphical example is shown in the appendix.

Definition 3. (Operational costs, refueling stops, and policy sets). For a $[l_s, l_t]$ refueling policy $\pi_{s\to t}^{l_s,l_t} = \langle (v_0, f_0), \ldots, (v_m, f_m)\rangle$ from vertex s to vertex t, we define the operational cost of the policy as $Cost(\pi_{s\to t}^{l_s,l_t}) = \sum_{1\leq i\leq m} P(v_i) * f_i$. We say v_i is a refueling vertex in $\pi_{s\to t}^{l_s,l_t}$ if and only if $f_i > 0$ and denote the set of refueling vertices from which the vehicle must stop to purchase fuel as $Refuelings(\pi_{s\to t}^{l_s,l_t}) = \{f_i | f_i > 0\}$. For a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$, we denote the set of all $[l_s, l_t]$ refueling policies from vertex s to vertex t as $\prod_{s\to t}^{l_s,l_t}$, and denote the set of all $[l_s, l_t]$ refueling policies from vertex s to vertex t using at most k refueling stops as $\prod(k)_{s\to t}^{l_s,l_t} = \{\pi | \pi \in \prod_{s\to t}^{l_s,l_t}, |Refuelings(\pi)| \leq k\}$.

Definition 4. (The optimal refueling policy problems) Given a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$, a $[l_s, l_t]$ refueling policy $\pi_{s\to t}^{l_s,l_t}$ from vertex s to vertex t is an optimal $[l_s, l_t]$ (k-stop-bounded) refueling policy from vertex s to vertex t if and only if $Cost(\pi_{s\to t}^{l_s,l_t}) \leq Cost(\pi)$ for every π in $\prod_{s\to t}^{l_s,l_t}$ (for every π in $\prod(k)_{s\to t}^{l_s,l_t}$). **(i)** The computational task of the single-pair $[l_s, l_t]$ optimal (k-stop-bounded) refueling policy problem is to determine an optimal $[l_s, l_t]$ (k-stop-bounded) refueling policy $\pi_{s\to t}^{l_s,l_t}$ from vertex s to vertex t given a pair of vertices s and t in V and the fuel levels $l_s, l_t \in [L, U]$. **(ii)** Given the initial fuel levels at the vertices, the computational task of the all-pairs $[*, L]$ optimal (k-stop-bounded) refueling policy problem is to determine the optimal (k-stop-bounded) refueling policies $\pi_{s\to t}^{l_s,L}$ where l_s is the given initial fuel level at vertex s for all pairs of vertices s and t ($s \neq t$) in V **(iii)** The computational task of the all-pairs $[L, L]$ optimal (k-stop-bounded) refueling policy problem is to determine the optimal $[L, L]$ (k-stop-bounded) refueling policies $\pi_{s\to t}^{L,L}$ for all pairs of vertices s and t ($s \neq t$) in V.

Remark: Later in the paper, we show that solutions to the optimal $[L, L]$ (k-stop-bounded) refueling policy problem can be used to efficiently determine solutions to the other optimal refueling policy problems. The k-stop-bounded versions of the problems arise in practice since each refueling stop takes time and at times the number of stops must be bounded above to ensure timely arrival. The all-pairs $[*, L]$ optimal (k-stop-bounded) refueling policy problem is the one studied by Khuller et al. [3].

3 Combinatorial Properties of Optimal $[L, L]$ Refueling Policies

In this section, we explore the combinatorial properties of optimal $[L, L]$ refueling policies and define the related terminology. In the next section, based on these combinatorial properties we develop polynomial-time algorithms for solving optimal refueling policy problems. Note that the definitions in the following all implicitly assume a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$ in the context and we use $\hat{X}(u, v)$ to denote the shortest distance from u to v with X as the underlying distance measure over the directed graph $G = (V, A)$ modelling the transportation network.

Definition 5. (Stop sequences and $\rho(\pi)$). The stop sequence $Stops(\pi)$ of a refueling policy $\pi = \langle (v_0, f_0), \ldots, (v_m, f_m)\rangle$ where $m \geq 1$ is the maximal subsequence of the operational path $\langle v_0, \ldots, v_m\rangle$ that contains the starting vertex $s = v_0$, the destination vertex $t = v_m$, and all the refueling vertices in between. We define $\rho(\pi) = |Refuelings(\pi) - \{s, t\}|$ and denote the stop sequence of the refueling policy π as $Stops(\pi) = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_{\rho(\pi)}}, v_{i_{\rho(\pi)+1}}\rangle$ where $i_0 = 0$, $v_{i_0} = v_0 = s$, $i_{\rho(\pi)+1} = m$, $v_{i_{\rho(\pi)+1}} = v_m = t$, and $\{v_{i_1}, \ldots, v_{i_{\rho(\pi)}}\} = Refuelings(\pi) - \{s, t\}$ with $0 = i_0 < i_1 < \ldots < i_{\rho(\pi)} < i_{\rho(\pi)+1} = m$ when $\rho(\pi) > 0$.

Definition 6. (The shortest-subpath property). Given a path $\langle v_0, \ldots, v_m \rangle$, we refer to $\langle v_j, \ldots, v_{j'} \rangle$ where $0 \le j < j' \le m$ as the subpath from v_j to $v_{j'}$ within the path. Given a refueling policy $\pi = \langle (v_0, f_0), \ldots, (v_m, f_m) \rangle$ with the stop sequence $Stops(\pi) = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_{\rho(\pi)}}, v_{i_{\rho(\pi)+1}} \rangle$ for a vehicle-network instance $I = \langle L, U, V, A, X, P \rangle$, we say π satisfies the shortest-subpath property if and only if for every k in $[0, \rho(\pi)]$, the subpath $\langle v_{i_k}, v_{i_k+1}, \ldots, v_{i_{k+1}} \rangle$ from v_{i_k} to $v_{i_{k+1}}$ within the operational path $\langle v_0, \ldots, v_m \rangle$ is a shortest path from vertex v_{i_k} to $v_{i_{k+1}}$ in the directed graph $G = (V, A)$ with the fuel consumption fuction X regarded as the distance measure.

Lemma 1. (The shortest-subpath property and optimality). Given a $[L, L]$ refueling policy $\pi^{L,L}_{s \to t}$ where $s \neq t$, if $\pi^{L,L}_{s \to t}$ does not satisfy the shortest-subpath property, then there exists a $[L, L]$ refueling policy $\widehat{\pi}^{L,L}_{s \to t}$ satisfying the shortest-subpath property such that $Cost(\widehat{\pi}^{L,L}_{s \to t}) \le Cost(\pi^{L,L}_{s \to t})$ and $\widehat{\pi}^{L,L}_{s \to t}$ has the same stop sequences $Stops(\widehat{\pi}^{L,L}_{s \to t}) = Stops(\pi^{L,L}_{s \to t})$.

Proof Sketch. Since no fuel is purchased between two refueling stops, the fuel prices at the vertices visited in between do not affect the total cost the refueling policy and it will always reduce cost by following a shortest path to reach from one refueling stop to the next refueling stop.

Definition 7. (The LU adjacency property). Given a refueling policy π with the stop sequence $Stops(\pi) = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_{\rho(\pi)}}, v_{i_{\rho(\pi)+1}} \rangle$, we say the refueling policy π satisfies the LU adjacency property if and only if for every k in $[0, \rho(\pi)]$, either $FL_\pi(v_{i_k}) = U$ or $FE_\pi(v_{i_{k+1}}) = L$, i.e. either the tank is full when leaving v_{i_k} or the tank is at the minimum fuel level required when arriving at the next refueling stop $v_{i_{k+1}}$.

Lemma 2. (The LU adjacency property and optimality). Given a $[L, L]$ refueling policy $\pi^{L,L}_{s \to t}$ where $s \neq t$, if $\pi^{L,L}_{s \to t}$ does not satisfy the LU adjacency property, then there exists a $[L, L]$ refueling policy $\widehat{\pi}^{L,L}_{s \to t}$ satisfying the LU adjacency property such that $Cost(\widehat{\pi}^{L,L}_{s \to t}) \le Cost(\pi^{L,L}_{s \to t})$, $\widehat{\pi}^{L,L}_{s \to t}$ has no more refueling vertices than $\pi^{L,L}_{s \to t}$, and $Stops(\widehat{\pi}^{L,L}_{s \to t})$ is a subsequence of $Stops(\pi^{L,L}_{s \to t})$.

Proof Sketch. Consider two adjacent refueling stops v_{i_k} and $v_{i_{k+1}}$. If $FL_\pi(v_{i_k}) \neq U$ and $FE_\pi(v_{i_{k+1}}) \neq L$, we can keep increasing the fuel purchased at the stop with the lower fuel price and decreasing the fuel purchased at the other stop without affecting the refueling decisions at other stops until either $FL_\pi(v_{i_k}) = U$ or $FE_\pi(v_{i_{k+1}}) = L$ or one of the two refueling stops can be eliminated from the stop sequence to reach a new reduced case with fewer refueling stops. Lemma 2 then follows by induction on the number of stops in the stop sequences.

Theorem 1. (Properties of optimal $[L, L]$ refueling policies). Every optimal $[L, L]$ (k-stop-bounded) refueling policy satisfies the shortest-subpath property and the LU adjacency property.

Proof sketch. It follows from Lemma 1 and Lemma 2.

Definition 8. (*LU* states, *LU* successors, *LU* transitions, and transition costs). Given a $[L, L]$ refueling policy $\pi = \pi^{L,L}_{s \to t} = \langle (v_0, f_0), \ldots, (v_m, f_m) \rangle$ satisfying the LU adjacency property and $s \neq t$ with the stop sequence $Stops(\pi) = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_{\rho(\pi)}}, v_{i_{\rho(\pi)+1}} \rangle$, we say (v_{i_k}, L) is an L state of π if $FE_\pi(v_{i_k}) = L$ and (v_{i_k}, U) is a U state of π if $FL_\pi(v_{i_k}) = U$. We refer to the L states and U states in π together as the LU states of π. We define a LU successor relation and four types of state transitions and their costs among LU states of π and refer to the transitions as the LU transitions of π as follows.
(i) Given an L state(v_{i_k}, L), (v_{i_k}, U) is an LU successor of (v_{i_k}, L) and $\langle (v_{i_k}, L), (v_{i_k}, U) \rangle$ is an $[L, U]$ transition in π with the cost $TC_\pi((v_{i_k}, L), (v_{i_k}, U))$ $= f_{i_k} * P(v_{i_k})$ if and only if $FE_\pi(v_{i_k}) = L$ and $FL_\pi(v_{i_k}) = U$.
(ii) Given an L state(v_{i_k}, L), $(v_{i_{k+1}}, L)$ is an LU successor of (v_{i_k}, L) and $\langle (v_{i_k}, L), (v_{i_{k+1}}, L) \rangle$ is a $[L, L]$ transition in π with the cost $TC_\pi((v_{i_k}, L),$ $(v_{i_{k+1}}, L)) = f_{i_k} * P(v_{i_k})$ if and only if $FE_\pi(v_{i_k}) = L$ and $FE_\pi(v_{i_{k+1}}) = L$.
(iii) Given a U state (v_{i_k}, U), $(v_{i_{k+1}}, U)$ is an LU successor of (v_{i_k}, U) and $\langle (v_{i_k}, U), (v_{i_{k+1}}, U) \rangle$ is a $[U, U]$ transition in π with the cost $TC_\pi((v_{i_k}, U),$ $(v_{i_{k+1}}, U)) = f_{i_{k+1}} * P(v_{i_{k+1}})$ if and only if $FL_\pi(v_{i_k}) = U$ and $FL_\pi(v_{i_{k+1}}) = U$.
(iv) Given a U state (v_{i_k}, U), $(v_{i_{k+2}}, L)$ is an LU successor of (v_{i_k}, U) and $\langle (v_{i_k}, U), (v_{i_{k+2}}, L) \rangle$ is a $[U, L]$ transition in π with the cost $TC_\pi((v_{i_k}, U),$ $(v_{i_{k+2}}, L)) = f_{i_{k+1}} * P(v_{i_{k+1}})$ if and only if $FL_\pi(v_{i_k}) = U$, and $FE_\pi(v_{i_{k+2}}) = L$. We refer to vertex $v_{i_{k+1}}$ above as the medium vertex of the $[U, L]$ transition.

Definition 9. (*LU* transition sequences). Given a $[L, L]$ refueling policy $\pi = \pi^{L,L}_{s \to t} = \langle (v_0, f_0), \ldots, (v_m, f_m) \rangle$ satisfying the LU adjacency property and $s \neq t$ with the stop sequence $Stops(\pi) = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_{\rho(\pi)}}, v_{i_{\rho(\pi)+1}} \rangle$, an LU state transition sequence of π is a representation of LU transitions from (s, L) to (t, L) as a sequence $\boldsymbol{LU}_\pi = \langle (v_{j_0}, l_0), \ldots, (v_{j_{\rho(\pi)+1}}, l_{\rho(\pi)+1}) \rangle$ of length $\rho(\pi) + 2$ where $l_i \in \{L, U\}$ for i in $[0, \rho(\pi) + 1]$, $j_0 = 0$, $v_{j_0} = v_0 = s$, $j_{\rho(\pi)+1} = m$, $v_{j_{\rho(\pi)+1}} = v_m = t$, $l_0 = L$, $l_{\rho(\pi)+1} = L$, $(v_{j_{k+1}}, l_{k+1})$ is the successor state of (v_{j_k}, l_k) for k in $[0, \rho(\pi)]$, and $0 = j_0 \leq j_1 \leq \ldots \leq j_{\rho(\pi)} < j_{\rho(\pi)+1} = m$ when $\rho(\pi) > 0$. We define the cost of an LU state transition sequence $\boldsymbol{LU}_\pi = \langle (v_{j_0}, l_0), \ldots, (v_{j_{\rho(\pi)+1}}, l_{\rho(\pi)+1}) \rangle$ as $Cost(\boldsymbol{LU}_\pi) = \sum_{0 \leq j \leq \rho(\pi)} TC_\pi((v_{j_k}, l_k), (v_{j_{k+1}}, l_{k+1}))$, i.e. the summation of the LU state transition costs between adjacent LU states in $\boldsymbol{LU}_\pi$.

Lemma 3. (Equality of the total *LU* transition cost and the total refueling cost). Given a $[L, L]$ refueling policy $\pi = \pi^{L,L}_{s \to t} = \langle (v_0, f_0), \ldots, (v_m, f_m) \rangle$ satisfying the LU adjacency property and $s \neq t$, the cost of every LU state transition sequence $\boldsymbol{LU}_\pi$ of π always equals the cost of the refuelling policy π, i.e. $Cost(\boldsymbol{LU}_\pi) = Cost(\pi)$, and the number of refueling stops in π equals the length of the transition sequence minus one.

Proof Sketch. Each LU transition involves exactly one refueling operation and the transition cost is exactly the cost of the refueling operation. Therefore the number of refueling stops in π equals the length of the transition sequence minus one. The sum of transition costs of the transition sequence therefore always equals the sum of the refueling costs, i.e. the total operation cost of the refueling policy. This is true even if in degenerate cases there may be multiple ways of interpreting π into LU transition sequences.

Definition 10. (Optimal LU transition paths). For the four types of LU transitions respectively, we define the following four types of optimal LU transition paths in the directed graph $G = (V, A)$:

- a path p is an optimal transition path for an $[L, U]$ transition $\langle(v_i, L), (v_i, U)\rangle$ if and only if p is the self loop $\langle v_i, v_i\rangle$,
- a path p is an optimal transition path for an $[L, L]$ transition $\langle(v_i, L), (v_j, L)\rangle$ where $v_i \neq v_j$ if and only if p is a shortest path from v_i to v_j in $G = (V, A)$,
- a path p is an optimal transition path for an $[U, U]$ transition $\langle(v_i, U), (v_j, U)\rangle$ where $v_i \neq v_j$ if and only if p is a shortest path from v_i to v_j in $G = (V, A)$, and
- a path p is an optimal transition path for an $[U, L]$ transition $\langle(v_i, U), (v_j, L)\rangle$ where $v_i \neq v_j$ if and only if $p = \langle v_i, \ldots, v_k, \ldots v_j\rangle$ where $v_i \neq v_k \neq v_j$, the subpath $\langle v_i, \ldots, v_k\rangle$ in p is a shortest path from v_i to v_k in $G = (V, A)$, the subpath $\langle v_k, \ldots, v_j\rangle$ in p is a shortest path from v_k to v_j in $G = (V, A)$, and $v_k = \arg\min_v(\hat{X}(v_i, v) + \hat{X}(v, v_j) - (U - L)) * P(v)$ over every vertex v in V where $\hat{X}(v_i, v) \leq (U - L))$, $\hat{X}(v, v_j) \leq (U - L))$, and $\hat{X}(v_i, v) + \hat{X}(v, v_j) > (U - L)$. We refer to vertex v_k above as the medium vertex of the optimal transition path for the $[U, L]$ transition.

Definition 11. (The optimal LU transition property). Given an $[L, L]$ refueling policy $\pi = \langle(v_0, f_0), \ldots, (v_m, f_m)\rangle$, we say π satisfies the optimal LU transition property if and only if (i) π satisfies both the shortest subpath property and the LU adjacency property and (ii) for every LU transition sequence $\boldsymbol{LU}_\pi = \langle(v_{j_0}, l_0), \ldots, (v_{j_{\rho(\pi)+1}}, l_{\rho(\pi)+1})\rangle$ of π and for every k in $[0, \rho(\pi)]$, the subpath $\langle v_{j_k}, v_{1+j_k} \ldots, v_{j_{k+1}}\rangle$ from v_{j_k} to v_{j_k+1} within the operational path $\langle v_0, , \ldots, v_m)\rangle$ is an optimal LU transition path in $G = (V, A)$.

Theorem 2. (More on the properties of optimal $[L, L]$ refueling policies). Every optimal $[L, L]$ (k-stop-bounded) refueling policy satisfies the optimal LU transition property.

Proof Sketch. By Theorem 1, every optimal $[L, L]$ (k-stop-bounded) refueling policy must satisfy both the shortest subpath property and the LU adjacency property. It follows then the transition paths (i.e. the sequences of vertices visited between two adjacent refueling stops) for $[L, L]$ transitions and $[U, U]$ must be the shortest paths between the adjacent refueling vertices. By Lemma 3, every optimal $[L, L]$ (k-stop-bounded) refueling policy must also have the minimal-cost $[U, L]$ transition between a pair of adjacent U state and an L state in the policy, which ensures that for an $[U, L]$ transition, the transition path (i.e. the sequences of vertices visited between two adjacent pairs of adjacent refueling stops) must have the optimal transition structure required by the optimal LU transition property.

Corollary 1. (The optimal LU transition costs). For every optimal $[L, L]$ (k-stop-bounded) refueling policy $\pi = \langle(v_0, f_0), \ldots, (v_m, f_m)\rangle$, the transition costs of LU transitions in π always have the following optimal-transition-cost structure:

- if $\langle(v_i, L), (v_i, U)\rangle$ is a $[L, U]$ transition in π, then the transition cost is $TC_\pi((v_i, L), (v_i, U)) = f_i * P(v_i) = (U - L) * P(v_i)$,
- if $\langle(v_i, L), (v_j, L)\rangle$ is a $[L, L]$ transition in π, then the transition cost is $TC_\pi((v_i, L), (v_j, L)) = f_i * P(v_i) = \hat{X}(v_i, v_j) * P(v_i)$,
- if $\langle(v_i, U), (v_j, U)\rangle$ is a $[U, U]$ transition in π, then the transition cost is $TC_\pi((v_i, U), (v_j, U)) = f_j * P(v_j) = \hat{X}(v_i, v_j) * P(v_j)$, and
- if $\langle(v_i, U), (v_j, L)\rangle$ is a $[U, L]$ transition in π, then the transition cost is $TC_\pi((v_i, U), (v_j, L)) = f_k * P(v_k) = (\hat{X}(v_i, v_k) + \hat{X}(v_k, v_j) - (U - L)) * P(v_k)$ where v_k is the medium vertex of an optimal transition path for the $[U, L]$ transition $\langle(v_i, U), (v_j, L)\rangle$.

Proof Sketch. The optimal transition costs above follow from Theorem 2, the definition of optimal LU transition paths in Definition 10, and the definition of LU transitions in Definition 8.

Theorem 3. (Reduction of finding optimal $[L, L]$ refueling policies into finding optimal LU transition sequences). Finding an optimal $[L, L]$ (k-stop-bounded) refueling policy from vertex s to vertex t is equivalent to finding a minimum-cost LU transition sequence (of no more than k transitions) from the LU state (s, L) to the LU state (t, L), in which the LU transition costs have the optimal-transition-cost structure depicted in Corollary 1.
Proof Sketch. It follows from Lemma 3 and Corollary 2.

Lemma 4. (Compact representation of optimal $[L, L]$ refueling policies). Given the information of all-pairs shortest paths in the network, for each pair of vertices s and t we can compactly represent an optimal $[L, L]$ (k-stop-bounded) refueling policy $\pi = \pi^{L,L}_{s \to t} = \langle(v_0, f_0), \ldots, (v_m, f_m)\rangle$ as the refueling stop sequence $Stops(\pi) = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_{\rho(\pi)}}, v_{i_{\rho(\pi)+1}}\rangle$ together with $\langle f_{i_0}, f_{i_1}, \ldots, f_{i_{\rho(\pi)}}\rangle$, the sequence of refueling amounts at v_{i_k}'s for $0 \leq k \leq \rho(\pi)$.

Proof Sketch. Since no refueling occurs between two refueling stops, it is redundant to record the refueling information of $f_i = 0$ for every vertex v_i that is not a refueling stop. And because every optimal $[L, L]$ (k-stop-bounded) refueling policy satisfies the shortest subpath property, it is redundant to record the vertices between two adjacent refueling stops since we can use any shortest path between them and still maintain the optimality of the policy.

Theorem 4. (Determining an $[L, L]$ refueling policy from an LU transition sequence). Given an LU transition sequence $\boldsymbol{LU}$ of length m of an optimal $[L, L]$ (k-stop-bounded) refueling policy, Algorithm 1 below can determine in $O(m)$ time the compact representation of an $[L, L]$ refueling policy π that satisfies the optimal LU transition property and has an LU transition sequence $\boldsymbol{LU}_\pi = \boldsymbol{LU}$. If two different $[L, L]$ refueling policies both satisfy the optimal LU transition property and both have $\boldsymbol{LU}$ as a LU transition sequence, then they both have the same total refueling cost.

Proof sketch. It takes linear time since it is a linear scan through the transition sequence and it takes constant time to process each LU transition. The way the LU transitions are processed from Theorem 2, the definition of optimal LU transition paths in Definition 10, the optimal LU transition costs described in Corollary 2, and the definition of LU transitions in Definition 8.

Algorithm 1. Transforming an optimal LU transition sequence into a compact representation of the corresponding optimal $[L, L]$ refueling policy as a sequence of refuleing stops and a sequence of refueling amounts.

Input: a vehicle-network instance $\langle L, U, V, A, X, P\rangle$, an LU transition sequence $\boldsymbol{LU}$ of an optimal $[L, L]$ (k-stop-bounded) refueling policy π, and for all pairs of vertices v_i and v_j, the information of the shortest distance $\hat{X}(v_i, v_j)$ and the medium vertex of an optimal $[U, L]$ transition path from v_i to v_j.

Output: the compact representation of the corresponding optimal policy π as the sequence of refueling stops $Stops(\pi) = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_{\rho(\pi)}}, v_{i_{\rho(\pi)+1}}\rangle$, and the corresponding refueling amount sequence $F_\pi = \langle f_{i_0}, f_{i_1}, \ldots, f_{i_{\rho(\pi)}}\rangle$.

Steps:

1. Start with empty sequences $Stops(\pi) = \langle\rangle$, $F_\pi = \langle\rangle$
2. Examine every pair of adjacent LU states in the LU transition sequence $\boldsymbol{LU}$ from the first pair to the last pair:

{
    if the pair is an $[L, U]$ transition $\langle(v_i, L),(v_i, U)\rangle$,
        append v_i into $Stops(\pi)$ and append the refueling amount $U-L$ into F_π;
    if the pair is an $[L, L]$ transition $\langle(v_i, L),(v_j, L)\rangle$,
        append v_i into $Stops(\pi)$ and
        append the refueling amount $\hat{X}(v_i, v_j) * P(v_i)$ into F_π;
    if the pair is an $[U, U]$ transition $\langle(v_i, U),(v_j, U)\rangle$,
        append v_j into $Stops(\pi)$ and
        append the refueling amount $\hat{X}(v_i, v_j) * P(v_j)$ into F_π;
    if the pair is an $[U, L]$ transition $\langle(v_i, U),(v_j, L)\rangle$ nd v_k is
        a a medium vertex of an optimal $[U, L]$ transition path from v_i to v_j,
        append v_k into $Stops(\pi)$ and append the refueling amount
        $(\hat{X}(v_i, v_k) + \hat{X}(v_k, v_j) - (U - L)) * P(v_k)$ into F_π.
}

4 Solving the Optimal Refueling Policy Problems

Based on these combinatorial properties explored in the previous section, in the following we show that finding optimal $[L, L]$ vehicle refueling policies in a transportation network G is equivalent to finding shortest paths in an optimal LU transition graph derived from G, which is essentially a finite automaton modelling all possible optimal LU transitions between vertices together with a distance measure representing the optimal LU transition costs. This leads to simple and efficient polynomial-time algorithms for solving the all-pairs $[L, L]$ (k-stop-bounded) optimal refueling policy problem. We then show ways to efficiently solve other optimal refueling policy problems based on the solutions to the all-pairs optimal (k-stop-bounded) $[L, L]$ vehicle refueling problem.

Definition 12. (The optimal LU transition graphs) The optimal LU transition graph for a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$ is a directed graph $G_I^{LU} = (V_I^{LU}, A_I^{LU})$ with a distance measure $X_I^{LU} : V_I^{LU} \times V_I^{LU} \rightarrow \mathbb{Z}^+ \cup \{0, \infty\}$ where

- the vertex set $V_I^{LU} = \{(v, L)|v \in V\} \cup \{(v, U)|v \in V\}$ is the set of all LU states,
- the set of directed edges $A_I^{LU} = \{\langle(v, L), (v, U)\rangle|v \in V\} \cup$ $\{\langle(v, L), (v', L)\rangle|v, v' \in V, \hat{X}(v, v') \leq U - L\} \cup$ $\{\langle(v, U), (v', U)\rangle|v, v' \in V, \hat{X}(v, v') \leq U - L\} \cup$ $\{\langle(v, U), (v', L)\rangle|v, v' \in V, \hat{X}(v, v') > U - L,$ $\exists v'' \in V, \hat{X}(v, v'') \leq U - L, \hat{X}(v'', v') \leq U - L\}$ is the set of all possible LU transitions between LU states, and
- the distance metric function X_I^{LU} encodes information of the optimal LU transition costs described in Corollary 1:
  (i) $X_I^{LU}(y, y) = 0$ for $y \in V_I^{LU}$,
  (ii) $X_I^{LU}(y, z) = \infty$ for $(y, z) \notin A_I^{LU}$ when $y \neq z$,
  (iii) $X_I^{LU}((v, L), (v, U)) = (U - L) * P(v)$ for an optimal $[L, U]$ transition $\langle(v, L), (v, U)\rangle \in A_I^{LU}$,
  (iv) $X_I^{LU}((v, L), (v', L)) = \hat{X}(v, v') * P(v)$ for an optimal $[L, L]$ transition $\langle(v, L), (v', L)\rangle \in A_I^{LU}$,
  (v)$X_I^{LU}((v, U), (v', U)) = \hat{X}(v, v') * P(v)$ for an optimal $[U, U]$ transition $\langle(v, U), (v', U)\rangle \in A_I^{LU}$, and
  (vi) $X_I^{LU}((v, U), (v', L)) = (\hat{X}(v, v'') + \hat{X}(v'', v') - (U - L)) * P(v'')$ where $v'' \in V$ is the medium vertex in an optimal transition path for the $[U, L]$ transition $\langle(v, U), (v', L)\rangle \in A_I^{LU}$.

Lemma 5. (Complexity of constructing the optimal LU transition graphs). Given a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$, the optimal LU transition graph $G_I^{LU} = (V_I^{LU}, A_I^{LU})$, the optimal LU transition paths, the medium vertices on optimal $[U, L]$ transition paths, and the distance measure X_I^{LU} can all be determined in $O(n^3)$ time where $n = |V|$ is the number of vertices in the network.

Proof Sketch. Applying the Floyd-Warshall algorithm for all-pairs shortest paths, we can determine the shortest distance $\hat{X}(u, v)$ between any pair of vertices in the transporation network $G = (V, A)$ in $O(n^3)$ time. The optimal LU transition paths are simply shortest paths in the cases of $[L, L]$, $[L, U]$, and $[U, U]$ transitions. In the case of $[U, L]$ transitions, the optimal LU transition paths and the medium vertices can be determined by examining all possible concatenations of two shortest paths with a common medium vertex as described in

Definition 10, which takes $O(n^3)$ time time all together for the n^2 pairs of $[U, L]$ transitions. The optimal transition costs as the metric X_I^{LU} can be determined in $O(n^3)$ time similarly as described in Corollary 1.

Theorem 5. (Reduction to the all-pairs shortest path problem). Given a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$, $\pi = \pi_{s\to t}^{L,L}$ is an optimal $[L, L]$ (k-stop-bounded) refueling policy from s to t if and only if an LU transition sequence $\boldsymbol{LU}_\pi$ of π is a shortest path from (s, L) to (t, L) (of length k or less) in the optimal LU transition graph G_I^{LU}. Finding all-pairs $[L, L]$ (k-stop-bounded) optimal refueling policies for a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$ is equivalent to finding all-pairs shortest paths (of length up to k) in the optimal LU transition graph G_I^{LU}.

Proof sketch. According to Theorem 2 and the definition of the optimal LU transition graph, each LU transition sequence $\boldsymbol{LU}_\pi$ of each $[L, L]$ refueling policy $\pi = \pi_{s\to t}^{L,L}$, $s \neq t$ that satisfies the optimal LU transition property uniquely represents a path $p = \boldsymbol{LU}_\pi = \langle (s, L), \ldots (t, L)\rangle$ from (s, L) to (t, L) in the optimal LU transition graph G_I^{LU}. By Lemma 3, a shortest path from (s, L) to (t, L) (of length k or less) in the optimal LU transition graph G_I^{LU} is also an LU transition sequence of an optimal $[L, L]$ (k-stop-bounded) refueling policy from s to t.

Corollary 2. (Complexity of the all-pairs optimal $[L, L]$ refueling policy problem). Algorithm 2 below can solve the all-pairs optimal $[L, L]$ refueling policy problem in $O(n^3)$ time and solve the all-pairs $[L, L]$ optimal k-stop-bounded refueling policy problem in $O(n^3 \log k)$ time where n is the number of vertices in the transportation network.

Proof Sketch. It follows from Theorem 5, Lemma 5, and the fact that the Floyd-Warshall algorithm takes $O(n^3)$ time while the distance-matrix repeated-squaring algorithm takes $O(n^3 \log k)$ time [2].

Algorithm 2. All-pairs $[L, L]$ optimal (k-stop-bounded) refueling policies.
Input: a vehicle-network instance $I = \langle L, U, V, A, X, P\rangle$, and
optionally a bound k on the maximal number of refueling stops allowed.
Output: compact representations of all-pairs optimal $[L, L]$ refueling policies
or all-pairs optimal $[L, L]$ k-stop-bounded refueling policies if k is given
as sequences of refueling stops and the corresponding refueling amounts.
Steps:
1. Determine the optimal LU transition graph G_I^{LU}, the medium vertices of optimal $[U, L]$ transition paths, and the distance measure X_I^{LU}.
2. If k is not given, determine all-pairs shortest paths in G_I^{LU}
using the Floyd-Warshall algorithm [2],
otherwise determine all-pairs shortest paths of length up to k
using the distance-matrix repeated-squaring algorithm [2].
3. View each path found in step 2 above as an LU transition sequence $\boldsymbol{LU}$ and
apply algorithm 1 to determine the corresponding optimal $[L, L]$
(k-stop-bounded) refueling policy π.

Corollary 3. (Reduction of the all-pairs $[*, L]$ optimal refueling policy problem). The solution to an instance of the all-pairs $[*, L]$ optimal refueling policy problem can be determined in $O(n^3)$ time given the solution to the all-pairs $[L, L]$ optimal refueling policy problem on the same instance.

Proof Sketch. To reduce it to the case where all vertices have the same initial fuel level L, for each vertex s that has an initial fuel level $l_s > L$ we add an additional virtual vertex s' that has a unit fuel price of zero, an initial fuel level of L, and can only directly reach s by consuming $U - l_s$ units of fuel. An optimal $[L, L]$ policy from s' then corresponds to an optimal $[l_s, L]$ policy from s. For each vertex s (out of n vertices), it then takes $O(n^2)$ additional time to update and LU transition graph to find the $[l_s, L]$ policies from s to other vertices.

Corollary 4. (Reduction of the single-pair optimal refueling policy problem). The solution to an instance of the single-pair optimal $[l_s, l_t]$ optimal refueling policy problem can be determined in $O(n^2)$ time given the solution to the all-pairs $[L, L]$ optimal refueling policy problem on the same instance.

Proof Sketch. To reduce it to the case where all vertices have both the initial fuel level and the final minimum fuel level equal to L, we can add an additional virtual vertex s' for s as described in the proof sketch for Corollary 3, and for t add an additional virtual vertex t' that is only directly reachable from t by consuming l_t units of fuel. An optimal $[L, L]$ policy from s' to t' then corresponds to an optimal $[l_s, l_t]$ policy from s to t. It then takes $O(n^2)$ additional time to update and LU transition graph to find an optimal $[l_s, l_t]$ policy from s to t.

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