Reasoning about Probabilities of Transitions and Observations
Part I: Basics of Using HMMs for Spelling Recognition

Hidden Markov Models (HMMs): In Rabiner’s tutorial paper on HMMs, Rabiner describes a HMM of $n$ hidden states \{$S_1, S_2, \ldots, S_n$\} and $m$ possible observations \{$v_1, v_2, \ldots, v_m$\} in terms of ($\pi$, $A$, $B$) where (i) $\pi$ is an $n$-by-$1$ vector encoding the initial-state probabilities, (ii) $A$ is an $n$-by-$n$ matrix encoding the state transition probabilities, and (iii) $B$ is an $n$-by-$m$ matrix encoding the observation probabilities. More specifically,
- $\pi = <\pi_1, \pi_2, \ldots, \pi_n>$ and each $\pi_i$ in $\pi$ represents the probability of having $S_i$ as the initial state.
- $A$ is an $n$-by-$n$ matrix and each element $a_{ij}$ in $A$ represents the transition probability from $S_i$ to $S_j$ (i.e. the probability of ending in $S_j$ as the next state when the current state is $S_i$).
- $B$ is an $n$-by-$m$ matrix and each element $b_{ij}$ in $B$ represents the probability of observing $v_j$ in state $S_i$ (i.e. the probability of observing in $v_j$ when the current state is $S_i$).

Spelling recognition and the spelling model: The diagram below depicts the transition probabilities of a spelling model regarding how a particular person $P$ may type a 3-character word $abc$ as depicted in our class handout on HMMs for spelling recognition. The information provided by the diagram correspond to the information encoded in vector $\pi$ and matrix $A$ in Rabiner’s tutorial paper on HMMs where Rabiner describes a HMM in terms of ($\pi$, $A$, $B$).

![Diagram](image-url)

Figure 1: The spelling model regarding the word $abc$ with the parameters $de_{fp} = 2$, $p_{repeat} = 0.2$, and $p_{moveOn} = 0.8$.
**Question #1:** There are 5 hidden states \( \{ S_1=I, S_2=a, S_3=b, S_4=c, S_5=F \} \). In other words, \( n=5 \) is the number of hidden states in this case. What is the contents of the corresponding vector \( \pi \) (as a 1 x 5 row vector)? What is the contents of the corresponding matrix \( A \) (as a 5 x 5 row matrix)?

**Spelling recognition and the keyboard model:**
In the following, let’s use a simplified 1-dimensional keyboard of only 4 keys a, b, c, d as depicted in the simplified example in our class handout on HMMs for spelling recognition. The information provided by such keyboard model corresponds to the information encoded in matrix \( B \) in Rabiner’s tutorial paper on HMMs where Rabiner describes a HMM in terms of \((\pi, A, B)\).

**A simplified example:** Consider the situation that \( p_{\text{miss}} = 0.1, p_{\text{hit}} = 0.9 \text{, and } d_{\text{freq}} = 2 \). If the one-dimensional keyboard only has 4 keys a, b, c, d (instead of the full 26 keys), the probabilities of typographic mistakes when trying to type a are

- \( Pr(\text{Char} = b|\text{State} = a) = 0.04 \),
- \( Pr(\text{Char} = c|\text{State} = a) = 0.02 \), and
- \( Pr(\text{Char} = d|\text{State} = a) = 0.04 \).

**Question #2:** We may observe any of the four possible characters each time the person tries to type one character given the 4-keys 1-dimensional keyboard. In the end of the However, for convenience, let’s add (i) one special observation \( \text{ReadyToType} \) dedicated solely to the special state \( S_1=I \) and (ii) one special observation \( \text{EndOfWord} \) dedicated solely to the special state \( S_5=F \). This is because we know the person is ready to start the typing process for the word when we are in the special starting state \( I \). Similarly we know it is the end of the typing process for the word when we are in the special final state \( F \). In other words, (i) the special state \( S_1=I \) is associated with the special observation \( \text{ReadyToType} \) with probability equal to 1 and (ii) the special state \( S_5=F \) is associated with the special observation \( \text{EndOfWord} \) with probability equal to 1. So we have 6 possible observations \( \{ v_1=\text{ReadyToType}, v_2=a, v_3=b, v_4=c, v_5=d, v_6=\text{EndOfWord} \} \). In other words, \( m=6 \) is the number of different observations we may encounter. What is the contents of the corresponding matrix \( B \) (as a 5 x 6 matrix)?
Reasoning about Probabilities of Transitions and Observations:
There are many possibilities regarding the process of state transitions and the resulting characters observed when the person tries to type the word \textit{abc} according to the spelling model and the keyboard model described above. For example, the person’s mind may go through the following state sequence \( Q: I \rightarrow a \rightarrow b \rightarrow b \rightarrow F \) and produces observation sequence \( O: \text{ReadyToType} \rightarrow b \rightarrow b \rightarrow d \rightarrow \text{EndOfWord} \), which corresponds to the resulting character string \textit{bbd} observed. How likely could this happen? Equations 12 to 15 on page 262 in Rabiner’s tutorial paper show how we can calculate the joint probability of going through a given state sequence \( Q \) and seeing a sequence of observations \( O \).

Question #3:
What is the probability of going through a given state sequence \( Q: I \rightarrow a \rightarrow b \rightarrow b \rightarrow F \) and seeing a sequence of observations \( O: \text{ReadyToType} \rightarrow b \rightarrow b \rightarrow d \rightarrow \text{EndOfWord} \)?

Reasoning about Probabilities of Observations:
As discussed above, there are many possibilities regarding the process of state transitions and the resulting characters observed when the person tries to type the word \textit{abc} according to the spelling model and the keyboard model described above. How likely could the person end in the character string \textit{bbd} (i.e. observation sequence \( O: \text{ReadyToType} \rightarrow b \rightarrow b \rightarrow d \rightarrow \text{EndOfWord} \))?

Note that any feasible state sequence \( Q: I \rightarrow ? \rightarrow ? \rightarrow ? \rightarrow F \) of length 5 may lead to the observation sequence. Equations 16 and 17 on page 262 in Rabiner’s tutorial paper tell us that we may (i) enumerate every possible state sequence \( Q \) and for each \( Q \) calculate the joint probability of going through the state sequence \( Q \) and seeing a sequence of observations \( O \) and (ii) sum up all joint probabilities in (i) as the result.

Question #4:
Enumerate and show all the possible state sequence \( Q: I \rightarrow ? \rightarrow ? \rightarrow ? \rightarrow F \).

Question #5:
What is the probability of seeing the character string \textit{bbd} (i.e. observation sequence \( O: \text{ReadyToType} \rightarrow b \rightarrow b \rightarrow d \rightarrow \text{EndOfWord} \)) when the person tries to type the word \textit{abc}?

Note: A more efficient algorithm for finding the answer to Question 5 is the forward algorithm described in pages 262–263 in Rabiner’s tutorial paper.