Discrete Structures: Homework 6

In a Christmas party, there are n guests coming to the party and each of them brings a gift for exchange. The host collects the gifts, randomly arranges the gifts in order, and in the end of the party, the host distributes the gifts one by one (according to the order) to guests. (If you like, you can imagine the guests line up one by one to receive the gifts according to the alphabetical order of their names.) Each guest then gets one gift back. The outcome is a bad arrangement if one or more of the guests gets the same gift he/she brought to the party. How likely would such a gift-exchange scheme end in a bad arrangement?

One way to get a sense of the likelihood when n is small is to conduct some empirical study. We'll do so in the class. For example, let's form several groups of students with n equal to 5 or 6 persons in each group, representing as the guests attending a Christmas party. Let's have each group repeat 10 gift-exchange experiments and count the number of times we see a bad arrangement. The ratio of the total number of bad arrangements over the total number of experiments then gives us a sense of the likelihood of getting a bad arrangement.

The empirical approach is hard to carry out when n is big. Instead, in the following we are going to conduct precise analysis about the likelihood using the principle of inclusion and exclusion.

Note that if there are n guests ($n \geq 1$), the host have $n! = n \times (n-1) \times \ldots \times 1$ different ways to arrange the gifts into some order for the n guests to pick up one by one according to the alphabetical order of their names. An arrangement of the gifts is a bad arrangement if one (or more) of the guests gets back the same gift he/she brought to the party; otherwise, it is a good arrangement.

Problem # 1. To make it concrete, let's consider the case $n = 4$ and let's refer to the guests (and their gifts too) as 1, 2, 3, and 4 respectively, then the arrangement of gifts in the order of 1243 is a bad arrangement since every guest gets his/her own gift back. As a contrast, the arrangement of gifts in the order of 2143 is a good arrangement since nobody gets his/her own gift back.

(1) Let U be the set of all possible gift arrangements. Explicitly write down U by listing all its members. Let B be the set of all bad gift arrangements. Explicitly write down B by listing all its members. Let G be the set of all good gift arrangements. Explicitly write down G by listing all its members. What is $|B|/|U|$, the ratio of the number of elements in B over the number of elements in U?

(2) For all i where $1 \leq i \leq 4$, let B_i be the set of all possible gift arrangements

in which guest i will get his/her own gift back. Explicitly write down B_1, B_2, B_3, and B_4 by explicitly listing all their members. How many such B_i's are there? What is $|B_i|$ (i.e. the number of elements in B_i) for each set B_i?

(3) For all i and j where $1 \leq i < j \leq 4$, let B_{ij} be the set of all possible gift arrangements in which guest i and j will both get their own gifts back. Explicitly write down each B_{ij} for all i and j where $1 \leq i < j \leq 4$ by explicitly listing all their members. How many such B_{ij}'s are there? What is $|B_{ij}|$ (i.e. the number of elements in B_{ij}) for each set B_{ij}?

(4) For all i, j, and k where $1 \leq i < j < k \leq 4$, let B_{ijk} be the set of all possible gift arrangements in which guest i, j, and k will all get their own gifts back. Explicitly write down each B_{ijk} for all i, j, and k where $1 \leq i < j < k \leq 4$ by explicitly listing all their members. How many such B_{ijk}'s are there? What is $|B_{ijk}|$ (i.e. the number of elements in B_{ijk}) for each set B_{ijk}?

(5) For all i, j, k, and l where $1 \leq i < j < k < l \leq 4$, let B_{ijkl} be the set of all possible gift arrangements in which guest i, j, k, and l will all get their own gifts back. Explicitly write down each B_{ijkl} for all i, j, k, and l where $1 \leq i < j < k < l \leq 4$ by explicitly listing all their members. How many such B_{ijkl}'s are there? What is $|B_{ijkl}|$ (i.e. the number of elements in B_{ijkl}) for each set B_{ijkl}?

(6) Examine more closely the notations defined above for Christmas party and the gift exchange scenario, and it is easy to discover that (i) B equals $B_1 \bigcup B_2 \bigcup B_3 \bigcup B_4$ (i.e. $B = \bigcup_{1 \leq i \leq 4} B_i$), and (ii) B_{ij} always equals $B_i \bigcap B_j$ for all i and j where $1 \leq i < j \leq 4$, and (iii) B_{ijk} always equals $B_i \bigcap B_j \bigcap B_k$ for all i, j, and k where $1 \leq i < j < k \leq 4$. and (iv) B_{ijkl} always equals $B_i \bigcap B_j \bigcap B_k \bigcap B_l$ for all i, j, k, and l where $1 \leq i < j < k < l \leq 4$. Therefore we can apply the inclusion-exclusion principle to determine $|B| = |\bigcup_{1 \leq i \leq 4} B_i|$. Do so to verify that $|B|$ determined this way is the same as what you found in question 1 of Problem #1 above.

Problem # 2. In Problem # 1, we determine $|B|$, the size of the set of all bad arrangements, (i.e. we count the number of bad arrangements) when the number of guests equals 4 ($n = 4$). You can generalize the analysis to determine $|B|$ for any number of guests n based on the principle of inclusion and exclusion. Please do so to show that given the Christmas party scenario with n guests

(1) $|B| = -\sum_{1 \leq i \leq n} (-1)^i \frac{n!}{i!}$ and

(2) $\lim_{n\to\infty} |B|/|U| = 1 - e^{-1} \simeq 0.62$.

Notes & Hints: In a previous semester, 52 out of the 81 experiments done by four groups (12/20, 17/20, 12/20, 11/21) are bad arrangements with $\frac{52}{81} \simeq 0.64$. Note that (i) $e^{-1} = \sum_{0\leq i}(-1)^i \frac{1}{i!}$ and (ii) $|B|/|U|$ is the likelihood of getting a bad arrangement when we randomly redistribute the gifts back to the guests. You need to apply the principle of inclusion and exclusion to $|B_1 \bigcup B_2 \bigcup B_3 \ldots \bigcup B_n|$ and complete the details of the following steps.

First,
$|B| = |B_1 \bigcup B_2 \bigcup B_3 \ldots \bigcup B_n|$
$= \ldots$
$= \sum_{1\leq i\leq n}(-1)^{i+1}\frac{n!}{i!}$
$= -\sum_{1\leq i\leq n}(-1)^i \frac{n!}{i!}$

Second,
$|B|/|U|$
$= \ldots$
$= (-\sum_{1\leq i\leq n}(-1)^i \frac{1}{i!})$

Thirdly,
given that $e^{-1} = \sum_{0\leq i}(-1)^i \frac{1}{i!}$, show
$1 - e^{-1} = 1 - \sum_{0\leq i\leq\infty}(-1)^i \frac{1}{i!}$
$= \ldots$
$= -\sum_{1\leq i\leq\infty}(-1)^i \frac{1}{i!}$

Finally,
$\lim_{n\to\infty} |B|/|U|$
$= \ldots$
$= 1 - e^{-1}$.