

## Discrete Structures: Homework 8

Given  $m = 4$  distinct presents and  $n = 3$  distinct guests, consider the following present distribution scenario. The host will give the presents out one by one and for each present the host simply randomly gives it to one of the guests. Therefore it is possible that a guest may receive no presents, one present, or multiple presents. In this scenario, we consider an outcome as a **bad** outcome if one or more of the guests gets no presents at all. Otherwise, it is a good outcome.

Let's refer to the 4 presents as  $p_1, p_2, p_3$  and  $p_4$  and the 3 guests as #1, #2, and #3 respectively. Then we can represent each possible outcome in the form of  $\langle i, j, k, l \rangle$  where  $1 \leq i, j, k, l \leq 3$  and  $\langle i, j, k, l \rangle$  indicates that the receivers of the 4 presents ( $p_1, p_2, p_3$  and  $p_4$  in order) are guests # $i$ , # $j$ , # $k$ , # $l$  respectively. For example,  $\langle 3, 3, 3, 1 \rangle$  represents the outcome where guest #3 gets the first three presents ( $p_1, p_2$ , and  $p_3$ ) while guest #1 gets the fourth present  $p_4$ . In other words,  $S = \{\langle i, j, k, l \rangle | 1 \leq i, j, k, l \leq 3\}$  is the sample space. Let's assume each possible outcome  $\langle i, j, k, l \rangle$  is equally likely and adopt the equiprobable probability space that assigns the same probability to each possible outcome. Answer the following questions.

- (1) What is  $|S|$ , the size of the sample space  $S$ ? What is  $Pr(\langle i, j, k, l \rangle)$  for any specific outcome  $\langle i, j, k, l \rangle$ ?
- (2) Consider the event  $E_1$  that guest #1 receives no presents at all. Please formally describe  $E_1$  as a subset of  $S$ , determine  $|E_1|$  (i.e. the size of  $E_1$ ), and determine  $Pr(E_1)$  (i.e. the probability of  $E_1$ ).
- (3) Consider the event  $E_{12} = E_1 \cap E_2$  that guests #1 and #2 receive no presents at all. Please formally describe  $E_{12}$  as a subset of  $S$ , determine  $|E_{12}|$  (i.e. the size of  $E_{12}$ ), and determine  $Pr(E_{12})$  (i.e. the probability of  $E_{12}$ ).
- (4) Consider the event  $E_{123} = E_1 \cap E_2 \cap E_3$  that guests #1, #2 and #3 receive no presents at all. Please formally describe  $E_{123}$  as a subset of  $S$ , determine  $|E_{123}|$  (i.e. the size of  $E_{123}$ ), and determine  $Pr(E_{123})$  (i.e. the probability of  $E_{123}$ ).
- (5) Generalizing the use of the notations above, let's use  $E_i$  ( $1 \leq i \leq n = 3$ ) to refer to the event that guest # $i$  receives no presents at all, use  $E_{ij}$  ( $1 \leq i, j \leq n = 3, i \neq j$ ) to refer to the event that guest # $i$  and # $j$  receive no presents at all, use  $E_{ijk}$  ( $1 \leq i, j, k \leq n = 3, i \neq j \neq k \neq i$ ) to refer to the event that guest # $i$ , # $j$ , and # $k$  receive no presents at all, and finally use  $E$  to refer to the event that one or more of the guests receive no presents in the end. It is easy to see that  $E = E_1 \cup E_2 \cup E_3$ . Please use the principle of inclusion and exclusion to determine  $|E|$  (i.e.

the size of  $E$ ) and then determine  $Pr(E)$  (i.e. the probability of  $E$ ).

- (6) Let  $X_1$  be the random variable recording the number of presents received by guest #1 in an outcome. What is  $Pr(X_1 = k)$  for  $0 \leq k \leq m = 4$ ? What is  $E(X_1)$  the expected value of  $X_1$ ? What is  $Var(X_1)$  the variance of  $X_1$ ?