Homework# 9

1. Use the binomial theorem to answer the following questions:
   (i) What is the coefficient of $x^3$ in the expansion of $(x+1)^{100}$?
   (ii) What is the coefficient of $x^{97}$ in the expansion of $(x+1)^{100}$?
   (iii) What is the coefficient of $x^3$ in the expansion of $(x+2)^{100}$?

2. Apply the binomial theorem to prove that for any given natural number $n$, $1 \leq n$, we have
   \[ 1 = C(n,1) - C(n,2) + C(n,3) - C(n,4) + \ldots + (-1)^{n+1} C(n,n) \]
   In other words, prove that
   \[ 1 - 1 \cdot \sum_{i \in \{1, \ldots, n\}} (-1)^i C(n,i) \]
   Hint: think about the expansion of $(1+1)^n$.

3. Use the equation in #2 above to prove the principle of inclusion and exclusion by showing: when we apply the principle of inclusion and exclusion to count the number of elements in the union of $m$ sets $A_1 \cup A_2 \cup \ldots A_m$, every element $x$ in the union is counted exactly once. Hint: for an element $x$ that appears in $n$ out of the $m$ sets $A_1, A_2, \ldots A_m$, show the inclusion and exclusion counting process will end up counting it once.

4. Apply the binomial theorem to prove that for any given natural number $n$, $1 \leq n$ and any given natural number $m$, $1 \leq m$, we have
   \[ m^0 \cdot C(n,0) + m^1 \cdot C(n,1) + m^2 \cdot C(n,2) + m^3 \cdot C(n,3) + \ldots + m^n \cdot C(n,n) = (m+1)^n \]

5. (i) For any given natural number $n$, $1 \leq n$ and any given natural number $i$, $1 \leq i \leq n$, shows that $i \cdot C(n,i)$ equals $n \cdot C(n-1,i-1)$.
   (ii) Based on the result in (i) to determine: for any given natural number $n$, what is the value of
   \[ 1 \cdot C(n,1) + 2 \cdot C(n,2) + 3 \cdot C(n,3) + \ldots + n \cdot C(n,n) \]
   In other words, what is the value of
   \[ \sum_{0 \leq i \leq n} i \cdot C(n,i) \]

6. (Optional bonus) Use the binomial theorem to prove that the following inequality is always true for any given non-negative integers $m$ and $n$:
   \[ (m+n)! \, / \, (m+n)^{m+n} \leq (m!/m^n) \, (n!/n^n), \]
   which is equivalent to proving
   \[ C(m+n, m) \leq (m+n)^{m+n} / (m^n n^n) \]

   This is actually one of the problems of the 2004 Putnam college math contest.