Notation: Given a directed graph of n vertices,

(i) we denote the vertices as v_1, $v_{2,\dots,}$ and v_n in the following,

(ii) we denote the length of the directed edge from v_i to v_j as w_{ij}, we have $w_{ii} = 0$ when $i=j$, and we have $w_{ij} = \infty$ when there is no directed edge from v_i to v_j.

(iii) we use **W** to refer to the n by n matrix where the element on the ith row and the jth column is w_{ij},

(iv) a directed path from v_i to v_j is a sequence of directed edges starting from v_i and leading to v_j in the end,

(v) the term *the length of a path* refers to the sum over the length of each individual directed edge in the path,

(vi) we use the term *the shortest path* from v_i to v_j to refer to a path with the minimal length among the paths from v_i to v_j, and the length of such a path is *the shortest distance* from v_i to v_j ,

(vii) we use d_{ij}^m to denote the minimal of the length of any directed path from v_i to v_j with no more than $1 \leq m$ directed edges in the path,

(viii) we use $\boldsymbol{D}^m$ to refer to the n by n matrix where the element on the ith row and the jth column is d_{ij}^m.

Property: Assuming that the length of a directed edge is always non-negative, a shortest path never has more than $n-1$ edges, and thus d_{ij}^{n-1} always equals *the shortest distance* from v_i to v_j. In other words, the matrix $\boldsymbol{D}^{n-1}$ contains all the information of shortest distances between vertices.

Bottom-up Computation of $D^{(n-1)}$

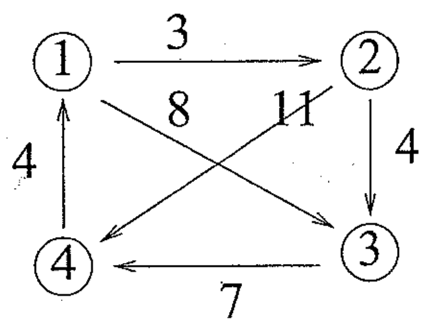
- Bottom: $D^{(1)} = \left[w_{ij}\right]$, the weight matrix.

- Compute $D^{(m)}$ from $D^{(m-1)}$, for $m = 2, ..., n-1$, using

$$d_{ij}^{(m)} = \min_{1 \leq k \leq n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}.$$

Example: Bottom-up Computation of $D^{(n-1)}$

Example

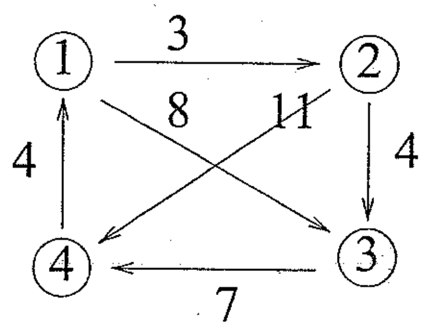


$D^{(1)} = [w_{ij}]$ is just the weight matrix:

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty \\ \infty & 0 & 4 & 11 \\ \infty & \infty & 0 & 7 \\ 4 & \infty & \infty & 0 \end{bmatrix}$$

Example: Computing $D^{(2)}$ from $D^{(1)}$

$$d_{ij}^{(2)} = \min_{1 \le k \le 4} \left\{ d_{ik}^{(1)} + w_{kj} \right\}.$$

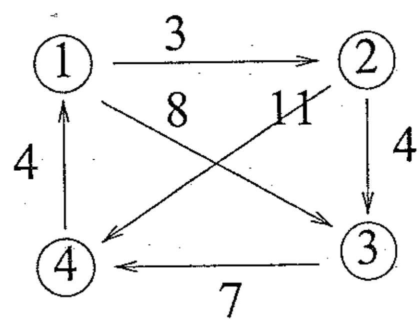


With $D^{(1)}$ given earlier and the recursive formula,

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 7 & 14 \\ 15 & 0 & 4 & 11 \\ 11 & \infty & 0 & 7 \\ 4 & 7 & 12 & 0 \end{bmatrix}$$

Example: Computing $D^{(3)}$ from $D^{(2)}$

$$d_{ij}^{(3)} = \min_{1 \le k \le 4} \left\{ d_{ik}^{(2)} + w_{kj} \right\}$$



With $D^{(2)}$ given earlier and the recursive formula,

$$D^{(3)} = \begin{bmatrix} 0 & 3 & 7 & 14 \\ 15 & 0 & 4 & 11 \\ 11 & 14 & 0 & 7 \\ 4 & 7 & 11 & 0 \end{bmatrix}$$

$D^{(3)}$ gives the distances between any pair of vertices.