About Rock-Paper-Scissors in Homework #7

1. There is no Nash equilibrium when the players are restricted to pure strategies only. Note that given any strategy profile <x, y> where x and y are pure strategies, <x, y> is never a Nash equilibrium as explained below.

a) If x≠y, the loser can always pick the third action z where x ≠ z ≠ y as the pure strategy to win the game. In this case, <x, y> is not a Nash equilibrium.

b) If x=y, there is another action z to beat x either player can pick that action z as the pure strategy to win the game. In this case, <x, y> is not a Nash equilibrium either.

2. There is a Nash equilibrium when players are allowed to have mixed strategies.

Proposition 1: For this particular game alone, a mixed strategy that uses only two of the three pure strategies (R, P, and S) cannot lead to a Nash equilibrium either**. (Why? Think over it.)**

Proposition 2: As a consequence of Proposition 1, each player needs to use all three pure strategies (R and P and S) in the mixed strategy adopted to reach a Nash equilibrium.

Proposition 3: Consider the mixed strategy (r, p, s=1-r-p) used by player #2 where r and p and q are the probabilities of using R(Rock), P(Paper), and S(Scissors) respectively. To reach a Nash equilibrium, these probabilities must be selected in a way such that Player#1's expected payoffs when Player#1 adopts the pure strategy R and P and S respectively are all equal while player #2 uses the mixed strategy (r, p, s=1-r-p).

Note that if Player#1's expected payoffs when Player#1 adopts the pure strategy R and P and S respectively are **not** all equal while player #2 uses the mixed strategy (r, p, s=1-r-p), then Player#1's best response will not be a mixed strategy that uses all three pure strategies, which will not lead to a Nash equilibrium according to Proposition 2**. (Why? Think over it.)**

Finding the mixed-strategy for Player #2:

Table below describes Player#1's expected payoffs when Player#1 uses a pure strategy while player #2 uses the mixed strategy (r, p, s=1-r-p).

Player#1's pure strategy	Player#2's mixed strategy (r, p, s=1-r-p)			Player#1's expected payoff when Player#1 uses a pure strategy while player #2 uses the mixed strategy
	r	p	1-r-p	

R	0	-2	3	$-2p+3-3r-3p\ = \mathbf{3-3r-5p}$
P	2	0	-1	$2r-1+r+p = \mathbf{3r+p-1}$
S	-3	1	0	$\mathbf{-3r+p}$

According to **Proposition 3**, the probabilities need to selected such that **the three expected payoffs are the same and thus we have 3-3r-5p = 3r+p-1 = -3r+p. Solving the equations, we get**

$$3r+p-1 = -3r+p \ \Rightarrow\ 6r = 1\ \Rightarrow \boldsymbol{r} = \frac{\mathbf{1}}{\mathbf{6}}$$

$$3-3r-5p = 3r+p-1 \Rightarrow 6r+6p = 4 \Rightarrow r+p = \frac{4}{6} \Rightarrow \frac{1}{6}+p = \frac{4}{6} \Rightarrow \boldsymbol{p} = \frac{\mathbf{1}}{\mathbf{2}}$$

$$s = 1-r-p = 1-\frac{1}{6}-\frac{1}{2} \Rightarrow \boldsymbol{s} = \frac{\mathbf{1}}{\mathbf{3}}$$

Therefore the mixed strategy for Player#2 needs to be $\left\{\frac{1}{6},\frac{1}{2},\frac{1}{3}\right\}$ to reach a Nash equilibrium.

Finding the mixed-strategy for Player #1:

When Player#2 sticks to this mixed strategy $\left\{\frac{1}{6},\frac{1}{2},\frac{1}{3}\right\}$ determined above, the expected payoff for Player#1 is always **0 = 3-3r-5p = 3r+p-1 = -3r+p** no matter what strategy (mixed or pure) Player#1 chooses.

However, using the exactly same logic reasoning, to reach a Nash equilibrium Player #1 also needs pick a mixed strategy such that that (i) that mixed strategy uses all three pure strategies and (ii) Player#2's expected payoff when Player#2 uses a pure strategy while player #1 uses that mixed strategy is always the same no matter which pure strategy Player#2 uses. Going through the exact same steps again, you'll find Player#1's mixed strategy should be $\left\{\frac{1}{6},\frac{1}{2},\frac{1}{3}\right\}$ too.

This is no surprise since this game is entirely symmetric for Player#1 and Player#2 and the equations you will go through are therefore exactly the same equations as you see for both players.

Finding the mixed-strategy Nash equilibrium:

Therefore < $\left\{\frac{1}{6},\frac{1}{2},\frac{1}{3}\right\}$, $\left\{\frac{1}{6},\frac{1}{2},\frac{1}{3}\right\}$ > is the mixed-strategy Nash equilibrium for this game. At that point, neither of the two players has an incentive to unilaterally change since the mixed strategy $\left\{\frac{1}{6},\frac{1}{2},\frac{1}{3}\right\}$ is a best response (actually every strategy pure or mixed is also a best response at this point) to the mixed strategy $\left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ to the other player.

Note that there are limits of the approach above for solving Rock Paper Scissors:

- **Propositions 1, 2, and 3 may not be true for** every zero-sum game**.**
- **Rock Paper Scissors** is symmetric for the two players, but not every zero-sum game is symmetric.

To see a linear-programming approach for finding Nash equilibria for zero-sum games, please read Sections 11.1~11.3 of *Linear Programming: Foundations and Extensions*, *2^{nd} ed*.