The simplex algorithm in a nutshell

1. **Canonical form for the simplex algorithm**: Given a linear program in standard form, add a slack variable into each inequality constraint to transform it into an equality constraint. This leads to the following canonical form for the simplex algorithm: Max c^TX subject to AX = b and X>=0 where A is a m-by-n matrix (n>m), b is a m-by-1 column vector, c is a n-by-1 column vector, and X is a n-by-1 column vector of n variables. **Consequently, the optimization problem is reduce to sub-problems regarding systems of linear equations.**

2. **Interior points, extreme points, and optimal solutions**: A point X in the feasible region F={X | AX = b, X>=0} is an interior point if X = αX' + βX'' for two distinct points X' and X'' in F where α>0, β>0, and α+β=1. Geometrically this means X is in the middle of the line segment connecting X' and X''. A point X in F is an extreme point if X is not an interior point. We have shown in the class that (i) there always exists an optimal solution on an extreme point if the linear program has a bounded optimal solution and (ii) any point X with more than m positive component values is an interior point, not an extreme point. **Consequently, we can find an optimal solution by examining the extreme points systematically.**

3. **Basic variables, basic feasible solutions, and extreme points**: A basic feasible solution is a solution to AX = b, X>=0 where (i) exactly m of the n variables are allowed to assume any non-negative values needed to satisfy the constraints AX = b, X>=0 while (ii) the values of the other n-m variables are restricted to 0. Those m variables are the basic variables corresponding to this basic feasible solution while the other n-m variables are non-basic variables. We have shown that each extreme point of the feasible region F={X | AX = b, X>=0} corresponds to a basic feasible solution in the context of m designate basic variables. **Consequently, we can find an optimal solution by examining the basic feasible solutions systematically, i.e. systematically examining ways of picking m basic variables.**

4. **Notation about the basis, coefficient vector of basic variables, and more:** In the context of a specific set of basic variables as the basis, **(i)** B denotes the matrix composed of the columns in A corresponding to the basic variables, **(ii)** c_B denotes the m-by-1 sub-vector from c recording the corresponding coefficients of the basic variables, and let **(iii)** Let X_B denote the m-by-1 vector recording the values of the basic variables in this basic feasible solution while the values of non-basic variables are 0.

5. **Big picture of the Simplex algorithm:** Starting from a basic feasible solution (an extreme point) given a set of m basic variables, iteratively examine the current basic feasible solution to search for a new set of m basic variables that lead to another basic feasible solution (an adjacent extreme point) to improve the objective function value. On each iteration (except for the last iteration), we look for a non-basic variable that can enter the basis to replace one of the current basic variables to form a new set of basic variables for the next iteration and improve the objective function value. An optimal solution is found when there is no way to improve the objective function value in this way further.

Algebraic perspective of the Simplex algorithm: Starting from a set of m basic variables for a basic feasible solution, do the following two steps repeatedly to move from one basic feasible solution to the next (i.e. from one extreme point to another extreme point) until we reach an optimal solution.

1. **Move to the basic feasible solution** corresponding to the set of basic variables:
   Transform the system of linear equations into $(B^{-1}A)X = B^{-1}b$. Since the values of all non-basic variables are 0, the values of the basic variables are determined by $X_B = B^{-1}b$. The objective function value attained at this point is $c_B^T(B^{-1}b)$.

2. **Check the optimality criterion $c_B^T(B^{-1}A) >= c^T$** and **search for possible improvement**:

   Three key notes about the optimality criterion:
   a) To increase the value of a non-basic variable x_j by 1 alone we need to decease the values of the basic variables by $(B^{-1}A_j)$ correspondingly in order to satisfy $(B^{-1}A)X = B^{-1}b$.
   b) Therefore $c_j - c_B^T(B^{-1}A_j)$ is the rate of change in the objective function value by increasing the value of a non-basic variable x_j corresponding to column A_j (while adjusting the values of the basic variables accordingly) where c_j is the coefficient of x_j in c.
   c) The maximum extent we can increase the value of a non-basic variable x_j corresponding to column A_j is the minimum among the ratios of elements in $(B^{-1}b)$ over the corresponding **positive** elements in A_j .

   Two possible outcomes:
   a) **A way found to move to the next basic feasible solution to improve the solution** if $c_B^T(B^{-1}A_j) < c_j$ for some non-basic variable x_j corresponding to column A_j: In this case, we can keep increasing the value of x_j (while adjusting the values of the basic variables accordingly) to improve the objective function value until the value of one of the basic variables become 0. **The maximum extent we can increase the value of such a non-basic variable x_j corresponding to column A_j is the minimum among the ratios of elements in $(B^{-1}b)$ over the corresponding positive elements in A_j**. If the i-th element in A_j leads to the minimum among the ratios of elements in $(B^{-1}b)$ over the corresponding positive elements in A_j, then x_j enters the basis while the i-th basic variable in the current basis leaves the basis, **leading to a new set of basic variables as a candidate for the next basic feasible solution**.
   b) **Optimal solution found** if $c_B^T(B^{-1}A) >= c^T$: In this case, there is no way to improve the objective function value by increasing the value of any of the non-basic variables (while adjusting the values of the basic variables accordingly) and optimality is attained at this point.

Tableau perspective of the Simplex algorithm: The algebraic operations described above can be efficiently carried out through a sequence of operations on the tableau:

1. Move to the basic feasible solution corresponding to the set of basic variables:

i. For the first iteration, nothing needs to be done if the slack variables are used as basic variables since B is the identity matrix in this case.

ii. For any other iteration later, some non-basic variables x_j corresponding to the *j*th column enters the basis to replace some *i*th basic variable in the previous basis. In this case, apply row operations to the tableau to transform the jth column into a unit vector where the i-th element is 1 while all the other elements are 0. This is equivalent to transforming the system of linear equations into $(B^{-1}A)X = B^{-1}b$ and keep the results of $B^{-1}A$ and $B^{-1}b$ explicitly in the tableau.

2. Check the optimality criterion $c_B^T(B^{-1}A) >= c^T$:

i. Since $B^{-1}A$ is explicitly kept in the tableau, $c_B^T(B^{-1}A)$ can be determined by the inner product of c_B with each column of $B^{-1}A$ now in the tableau.

ii. **If** $c_B^T(B^{-1}A_j) < c_j$ for some non-basic variable x_j corresponding to column A_j, we can keep increasing the value of x_j (while adjusting the values of the basic variables accordingly) to improve the objective function value until the value of one of the basic variables become 0. **The maximum extent we can increase the value of such a non-basic variable x_j corresponding to column A_j is the minimum among the ratios of elements in ($B^{-1}b$) over the corresponding positive elements in A_j**. If the i-th element in A_j leads to the minimum among the ratios of elements in ($B^{-1}b$) over the corresponding positive elements in A_j, then x_j enters the basis while the i-th basic variable in the current basis leaves the basis, **leading to a new set of basic variables as a candidate for the next basic feasible solution**.

iii. **Otherwise**, $c_B^T(B^{-1}A) >= c^T$ and there is no way to improve the objective function value by increasing the value of any of the non-basic variables (while adjusting the values of the basic variables accordingly). Optimality is attained at this point.

Illustration of the simplex algorithm through an example:

(Note this is exactly the sample example examined in Section 2.1 in *Linear Programming: Foundations and Extensions*)

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Max 5*X1 + 4*X2 + 3*X3

Subject to

X1	X2	X3		
2	3	1	<=	5
4	1	2	<=	11
3	4	2	<=	8

X1, X2, X3 >=0

****

Add slack variables, W1, W2, W3:

➔

Max 5*X1 + 4*X2 + 3*X3 + 0*W1 + 0*W2 + 0*W3

Subject to

X1	X2	X3	W1	W2	W3		
2	3	1	1			=	5
4	1	2		1		=	11
3	4	2			1	=	8

X1, X2, X3, W1, W2, W3 >=0

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Step 1A: Initial basic feasible solution:

Let W1, W2, W3 be the basic variables to form the basis **B** composed of the 3 columns corresponding to W1, W2, W3, which happens to be an identity matrix

➔Initial basic feasible solution W1=5, W2=11, W3=8 while X1=0, X2=0, X3=0

Objective function value attained: 5*X1 + 4*X2 + 3*X3=0

X1	X2	X3	**W1**	**W2**	**W3**		
2	3	1	**1**			=	**5**
4	1	2		**1**		=	**11**
3	4	2			**1**	=	**8**

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Step 1B: Evaluate non-basic variables to search for the next basic feasible solution that improves the objective function value attained:

Need to evaluate each of the non-basic variables: X1, X2, X3 where c_B^T = [0, 0, 0]

Evaluate X1:

	Coefficients of the objective function							
	X1	X2	X3	**W1**	**W2**	**W3**		
	5	4	3	**0**	**0**	**0**		

	X1	X2	X3	**W1**	**W2**	**W3**		
C W1: 0	**2**	3	1	**1**			=	**5**
C W2: 0	**4**	1	2		**1**		=	**11**
C W3: 0	**3**	4	2			**1**	=	**8**

To increase X1 by 1 unit, we need to decrease W1, W2, W3 accordingly by 2, 4, 3 (as shown in $B^{-1}A_1$) correspondingly. Since min((**5**,**11**, **8**)/(**2**, **4**, **3**)) = 5/2 ➔ 5/2 is the maximum allowed to increase while decreasing W1, W2, W3 accordingly and keeping them non-negative. When this happen, the non-basic variable X1 enters while the basic variable W1 leaves, leading to a new set of basic variables X1, W2, W3.

$c_1 - c_B^T(B^{-1}A_1)$ = 5-(**0**, **0** ,**0**).(**2**, **4**, **3**)= **5** ➔ Rate of increase of objective function value when increasing X1 while decreasing W1, W2, W3 accordingly:

Evaluate X2:

	Coefficients of the objective function							
	X1	**X2**	X3	**W1**	**W2**	**W3**		
	5	**4**	3	**0**	**0**	**0**		

	X1	**X2**	X3	**W1**	**W2**	**W3**		
C W1: 0	2	**3**	1	**1**			=	**5**
C W2: 0	4	**1**	2		**1**		=	**11**
C W3: 0	3	**4**	2			**1**	=	**8**

min((**5,11, 8**)/(**3**, **1**, **4**)) = 5/3 ➔ Maximum allowed to increase X2 while decreasing W1, W2, W3 accordingly and keeping them non-negative

$c_2 - c_B^T(B^{-1}A_2)$ = 4-(**0, 0 ,0**).(**3**, **1**, **4**) = 4 ➔ Rate of increase of objective function value when increasing X2 while decreasing W1, W2, W3 accordingly:

Evaluate X3:

	Coefficients of the objective function							
	X1	X2	**X3**	**W1**	**W2**	**W3**		
	5	4	**3**	**0**	**0**	**0**		

	X1	X2	**X3**	**W1**	**W2**	**W3**		
C W1: 0	2	3	**1**	**1**			=	**5**
C W2: 0	4	1	**2**		**1**		=	**11**
C W3: 0	3	4	**2**			**1**	=	**8**

min((**5,11, 8**)/(**1**, **2**, **2**)) = 8/2 🡐 Maximum allowed to increase X3 while decreasing W1, W2, W3 accordingly and keeping them non-negative

$c_3 - c_B^T(B^{-1}A_3)$= 3-(**0, 0 ,0**).(***1**, **2**, **2***) = 3 🡐 Rate of increase of objective function value when increasing X3 while decreasing W1, W2, W3 accordingly:

Findings: Any one of X1, X2, X3 can enter to improve the objective function value.

Let's pick X1 as a new basic variable to enter the basis and maximally increase it by 5/2 while decreasing W1, W2, and W3 accordingly, leading W1 to reach 0 and leave the basis to becoe a non-basic variable.

In other words, Let X1 replace the role of W1 to form the new basis.

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Step 2A: Update the tableau based on the findings in Step 1B to reach the next basic feasible solution that improves the objective function value attained:

Let X1, W2, W3 be the basic variables to form the new basis B composed of the 3 columns corresponding to X1, W2, W3:

X1	X2	X3	W1	**W2**	**W3**		
2	3	1	**1**			=	**5**
4	1	2		**1**		=	**11**
3	4	2			**1**	=	**8**

Transform the tableau above by applying the linear transformation of B^{-1} to the tableau, which is equivalent to applying row operations to make the columns corresponding to X1, W2, W3 an identity matrix

➔

X1	X2	X3	W1	**W2**	**W3**		
1	1.5	0.5	0.5			=	**2.5**
	-5	0	-2	**1**		=	**1**
	-0.5	0.5	-1.5		**1**	=	**0.5**

New basic feasible solution X1=2.5, W2=1, W3=0.5 while W1=0, X2=0, X3=0

Objective function value attained: 5*X1 + 4*X2 + 3*X3=**12.5**

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Step 2B: Evaluate non-basic variables to search for the next basic feasible solution that improves the objective function value attained:

Need to evaluate each of the non-basic variables: X2, X3, W1

Evaluate X2:

	Coefficients of the objective function							
	X1	X2	X3	W1	**W2**	**W3**		
	5	4	3	0	**0**	**0**		

	X1	X2	X3	W1	**W2**	**W3**		
C X1: 5	**1**	1.5	0.5	0.5			=	**2.5**
C W2: 0		-5	0	-2	**1**		=	**1**
C W3: 0		-0.5	0.5	-1.5		**1**	=	**0.5**

min((**2.5, ,**)/(**1.5, ,**)) = 5/3 ➔ Maximum allowed to increase X2 while decreasing X1, W2, W3 accordingly and keeping them non-negative

$c_2 - c_B^T(B^{-1}A_2)$ = 4-(**5, 0 ,0**).(**1.5**, **-5**, **-0.5**)= **-3.5** ➔ Rate of increase of objective function value when increasing X2 while decreasing X1, W2, W3 accordingly:

Evaluate X3:

	Coefficients of the objective function							
	X1	X2	X3	W1	**W2**	**W3**		
	5	4	3	0	**0**	**0**		

	X1	X2	X3	W1	**W2**	**W3**		
C X1: 5	**1**	1.5	0.5	0.5			=	**2.5**
C W2: 0		-5	0	-2	**1**		=	**1**
C W3: 0		-0.5	0.5	-1.5		**1**	=	**0.5**

min((**2.5, , 0.5**)/(**0.5, , 0.5**)) = 1 ➔ Maximum allowed to increase X3 while decreasing X1, W2, W3 accordingly and keeping them non-negative.

$c_3 - c_B^T(B^{-1}A_3)$ = 3-(**5, 0 ,0**).(**0.5**, **0**, **0.5**)= **0.5** ➔ Rate of increase of objective function value when increasing X3 while decreasing X1, W2, W3 accordingly:

Evaluate W1:

	Coefficients of the objective function							
	X1	X2	X3	W1	**W2**	**W3**		
	5	4	3	0	**0**	**0**		

	X1	X2	X3	W1	**W2**	**W3**		
C X1: 5	**1**	1.5	0.5	0.5			=	**2.5**
C W2: 0		-5	0	-2	**1**		=	**1**
C W3: 0		-0.5	0.5	-1.5		**1**	=	**0.5**

min((**2.5, ,**)/(**0.5, ,**)) = 5 ➔ Maximum allowed to increase W1 while decreasing X1, W2, W3 accordingly and keeping them non-negative.

$c_{W1} - c_B^T(B^{-1}A_{W1})$ = 0-(**5, 0 ,0**).(**0.5**, **-2**, **1.5**)= **-2.5** ➔ Rate of increase of objective function value when increasing W1 while decreasing X1, W2, W3 accordingly:

Findings: Only X3 can enter to improve the objective function value.

Let's pick X3 as a new basic variable to enter the basis and maximally increase it by 1 while decreasing X1, W2, and W3 accordingly, leading W3 to reach 0 and leave the basis to become a non-basic variable.

In other words, Let X3 replace the role of W3 to form the new basis.

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Step 3A: Update the tableau based on the findings in Step 2B to reach the next basic feasible solution that improves the objective function value attained:

Let X1, W2, X3 be the basic variables to form the new basis B composed of the 3 columns corresponding to X1, W2, X3:

X1	X2	**X3**	W1	**W2**	W3		
1	1.5	**0.5**	0.5			=	**2.5**
	-5	**0**	-2	**1**		=	**1**
	-0.5	**0.5**	-1.5		1	=	**0.5**

Transform the tableau above by applying the linear transformation of B^{-1} to the tableau, which is equivalent to applying row operations to make the columns corresponding to X1, W2, X3 an identity matrix

➔

X1	X2	X3	W1	W2	W3		
1	2		2		-1	=	**2**
	-5		-2	**1**	0	=	**1**
	-1	**1**	-3		2	=	**1**

New basic feasible solution X1=2, W2=1, X3=1 while X2=0, W1=0, W3=0

Objective function value attained: 5*X1 + 4*X2 + 3*X3=13

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Step 3B: Evaluate non-basic variables to search for the next basic feasible solution that improves the objective function value attained:

Need to evaluate each of the non-basic variables: X2, W1, W3

Evaluate X2:

	Coefficients of the objective function						
	X1	X2	**X3**	W1	**W2**	W3		
	5	4	**3**	0	**0**	0		

	X1	X2	X3	W1	W2	W3		
C X1: 5	**1**	2		2		-1	=	**2**
C W2: 0		-5		-2	**1**	0	=	**1**
C X3: 3		-1	**1**	-3		2	=	**1**

min((**2, ,**)/(**2, ,**)) = 1→ Maximum allowed to increase X2 while decreasing X1, W2, X3 accordingly and keeping them non-negative.

$c_2 - c_B^T(B^{-1}A_2)$ = 4-(**5, 0 ,3**).(**2**, **-5**, **-1**)= **-3** → Rate of increase of objective function value when increasing X2 while decreasing X1, W2, X3 accordingly.

Evaluate W1:

	Coefficients of the objective function						
	X1	X2	**X3**	W1	**W2**	W3		
	5	4	**3**	0	**0**	0		

	X1	X2	**X3**	W1	**W2**	W3		
C X1: 5	**1**	2		2		-1	=	**2**
C W2: 0		-5		-2	**1**	0	=	**1**
C X3: 3		-1	**1**	-3		2	=	**1**

min((**2, ,**)/(**2, ,**)) = 1➔ Maximum allowed to increase W1 while decreasing X1, W2, X3 accordingly and keeping them non-negative.

$c_{W1}-c_B^T(B^{-1}A_{W1})$=0-(**5, 0** ,**3**).(**2**, **-2**, **-3**)= **-1** ➔ Rate of increase of objective function value when increasing W1 while decreasing X1, W2, X3 accordingly.

Evaluate W3:

	Coefficients of the objective function							
	X1	X2	**X3**	W1	**W2**	W3		
	5	4	**3**	0	**0**	0		

	X1	X2	**X3**	W1	**W2**	W3		
C X1: 5	**1**	2		2		-1	=	**2**
C W2: 0		-5		-2	**1**	0	=	**1**
C X3: 3		-1	**1**	-3		2	=	**1**

min((**, , 1**)/(**, , 2**)) = 0.5➔ Maximum allowed to increase W3 while decreasing X1, W2, X3 accordingly and keeping them non-negative

$c_{W3}-c_B^T(B^{-1}A_{W3})$=0-(**5, 0** ,**3**).(**-1**, **0**, **2**)= **-1** ➔ Rate of increase of objective function value when increasing W3 while decreasing X1, W2, X3 accordingly.

Findings: None of the non-basic variable can enter to improve the objective function value.

➔ Optimal solution reached.

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Exercise: Try the simplex algorithm on the following case:

Max 15*X1 + 8*X2

Subject to

X1	X2		
3	4	<=	12
5	2	<=	10

X1, X2 >=0

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Add slack variables, W1, W2:

➔

Max 15*X1 + 8*X2 + 0*W1 + 0*W2

Subject to

X1	X2	W1	W2		
3	4	1	0	=	12
5	2	0	1	=	10

X1, X2, W1, W2 >=0

****

Step 1A: Initial basic feasible solution:

Let W1, W2 be the basic variables to form the basis **B** composed of the 2 columns corresponding to W1, W2 which happens to be an identity matrix: