About Rock-Paper-Scissors in Homework #7

1. There is no Nash equilibrium when the players are restricted to pure strategies only. Note that given any strategy profile \(<x, y>\) where x and y are pure strategies, \(<x, y>\) is never a Nash equilibrium as explained below.
   
a) If \(x \neq y\), the loser can always pick the third action z where \(x \neq z \neq y\) as the pure strategy to win the game. In this case, \(<x, y>\) is not a Nash equilibrium.

b) If \(x = y\), there is another action z to beat x either player can pick that action z as the pure strategy to win the game. In this case, \(<x, y>\) is not a Nash equilibrium either.

2. There is a Nash equilibrium when players are allowed to have mixed strategies.

**Proposition 1**: For this particular game alone, a mixed strategy that uses only two of the three pure strategies (R, P, and S) cannot lead to a Nash equilibrium either. *(Why? Think over it.)*

**Proposition 2**: As a consequence of Proposition 1, each player needs to use all three pure strategies (R and P and S) in the mixed strategy adopted to reach a Nash equilibrium.

**Proposition 3**: Consider the mixed strategy \((r, p, s=1-r-p)\) used by player #2 where r and p and q are the probabilities of using R(Rock), P(Paper), and S(Scissors) respectively. To reach a Nash equilibrium, these probabilities must be selected in a way such that Player #1’s expected payoffs when Player #1 adopts the pure strategy R and P and S respectively are all equal while player #2 uses the mixed strategy \((r, p, s=1-r-p)\).

Note that if Player #1’s expected payoffs when Player #1 adopts the pure strategy R and P and S respectively are not all equal while player #2 uses the mixed strategy \((r, p, s=1-r-p)\), then Player #1’s best response will not be a mixed strategy that uses all three pure strategies, which will not lead to a Nash equilibrium according to Proposition 2. *(Why? Think over it.)*

**Finding the mixed-strategy for Player #2:**

Table below describes Player #1’s expected payoffs when Player #1 uses a pure strategy while player #2 uses the mixed strategy \((r, p, s=1-r-p)\).

<table>
<thead>
<tr>
<th>Player #1’s pure strategy</th>
<th>Player #2’s mixed strategy ((r, p, s=1-r-p))</th>
<th>Player #1’s expected payoff when Player #1 uses a pure strategy while player #2 uses the mixed strategy</th>
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</table>
According to Proposition 3, the probabilities need to be selected such that the three expected payoffs are the same and thus we have $3 - 3r - 5p = 3r + p - 1 = -3r + p$. Solving the equations, we get

$$3r + p - 1 = -3r + p \Rightarrow 6r = 1 \Rightarrow r = \frac{1}{6}$$

$$3 - 3r - 5p = 3r + p - 1 \Rightarrow 6r + 6p = 4 \Rightarrow r + p = \frac{4}{6} \Rightarrow \frac{1}{6} + p = \frac{4}{6} \Rightarrow p = \frac{1}{2}$$

$$s = 1 - r - p = 1 - \frac{1}{6} - \frac{1}{2} \Rightarrow s = \frac{1}{3}$$

Therefore the mixed strategy for Player#2 needs to be $\left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ to reach a Nash equilibrium.

**Finding the mixed-strategy for Player #1:**

When Player#2 sticks to this mixed strategy $\left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ determined above, the expected payoff for Player#1 is always $0 = 3 - 3r - 5p = 3r + p - 1 = -3r + p$ no matter what strategy (mixed or pure) Player#1 chooses.

However, using the exactly same logic reasoning, to reach a Nash equilibrium Player #1 also needs to pick a mixed strategy such that (i) that mixed strategy uses all three pure strategies and (ii) Player#2’s expected payoff when Player#2 uses a pure strategy while player #1 uses that mixed strategy is always the same no matter which pure strategy Player#2 uses. Going through the exact same steps again, you’ll find Player#1’s mixed strategy should be $\left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ too.

This is no surprise since this game is entirely symmetric for Player#1 and Player#2 and the equations you will go through are therefore exactly the same equations as you see for both players.

**Finding the mixed-strategy Nash equilibrium:**

Therefore $\left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ is the mixed-strategy Nash equilibrium for this game. At that point, neither of the two players has an incentive to unilaterally change since the mixed strategy $\left\{\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right\}$ is a
best response (actually every strategy pure or mixed is also a best response at this point) to the mixed strategy \( \left( \frac{1}{6}, \frac{1}{2}, \frac{1}{3} \right) \) to the other player.

**Note that there are limits of the approach above for solving Rock Paper Scissors:**

- **Propositions 1, 2, and 3 may not be true for** every zero-sum game.

- **Rock Paper Scissors** is symmetric for the two players, but not every zero-sum game is symmetric.

**To see a linear-programming approach for finding Nash equilibria for zero-sum games,** please read Sections 11.1~11.3 of *Linear Programming: Foundations and Extensions, 2nd ed.*